Chapter 4. The solution of cubic and quartic equations

In the 16th century in Italy, there occurred the first progress on polynomial equations beyond the quadratic case. The person credited with the solution of a cubic equation is Scipione del Ferro (1465-1526), who lectured in arithmetic and geometry at the University of Bologna from 1496 until 1526. He wrote down a solution of the cubic equation in a manuscript, which passed to his student Annibale dalla Nave. The manuscript is now lost, and no work of del Ferro has survived. News of the solution spread by word of mouth and reached the mathematician Niccolò Tartaglia (whose surname means 'The Stammerer': he had difficulty in speaking after he received a sword cut to the tongue during a French siege of Brescia, where he lived). Girolamo Cardano heard that Tartaglia was in possession of the solution and he implored him to share it. Tartaglia eventually told Cardano the solution but swore him to secrecy. Later, Nave told Cardano of the existence of del Ferro's manuscript on the solution of the cubic and thereafter Cardano felt no longer bound by the terms of his oath to Tartaglia, as Tartaglia was not the originator of the method.

Cardano (1501-76) was an important figure in the development of early modern science, and was eager to hear of new developments, such as the solution of the cubic equation. He was also a famous physician, whose skills were sought throughout Europe. He wrote several books, but for our purposes, the most important is *Artis magnae, sive de regulis algebraicis* (Of the great art, or concerning the algebraic rules), better known as *Ars magna*, which was published in Nuremberg in 1545. Cardano is credited as the originator of the modern theory of algebraic equations, for he presented here for the first time to the public the solution of the cubic equation, as he had learnt it from Tartaglia. It is important to realize that work on algebra was hindered by lack of good notation, which only began to appear in the 17th century in France. It is not possible for the novice to read Cardano's work in its original form, as the notation is so unfamiliar, but a translation into modern English with standard notation is now available.

It was in his solution of the cubic equation that Cardano first encountered what we have come to know as complex numbers, and he may to some extent be credited with their discovery. He was, however, apparently unconvinced about the validity of the use of these imaginary or sophistic numbers. It might be added that there was a general reserve about using negative numbers at this time, and Cardano often looked at different forms of equations to avoid introducing negative numbers (writing them down in what we would consider to be repetitive but apparently different ways, so that all coefficients were positive). Nonetheless, the need for negative numbers must have been appreciated, for otherwise the complex numbers would never have appeared. It might be added that anxiety about negative numbers (not to mention imaginary numbers) was felt well into the 19th century.

Let us see what was involved in Cardano's solution. We will depart from his notation, but his basic idea remains the same and has not been improved to this day. We take a monic cubic polynomial

$$f(x) = x^3 + ax^2 + bx + c,$$

where a, b and c are the coefficients, which we can take to be real numbers, and we look for a root of this polynomial: this is a number α (which might turn out not to be real!) satisfying $f(\alpha) = 0$. The first thing that Cardano did was to change variables so as to form a cubic in a new variable y with the property that the coefficient of y^2 is 0. We do this as follows: Let $y = x + r$, where r is some number to be chosen, and evaluate $f(x) = f(y - r)$ in terms of y. We get

$$\begin{aligned} f(x) = f(y - r) &= y^3 - 3ry^2 + 3r^2y - r^3 + a(y^2 - 2ry + r^2) + b(y - r) + c \\ &= y^3 + (a - 3r)y^2 + dy + e, \end{aligned}$$

where d and e are two new coefficients (which we will not write down explicitly). We can see that if we choose r so that $a - 3r = 0$, the coefficient of y^2 is 0. This trick is used quite frequently in polynomial theory (trying to make several coefficients become zero by transformation of variables). It was extended by Tschirnhaus in the 17th century. For the purposes of finding the roots of a cubic polynomial, this transformation shows that it will suffice to be able to find the roots of one of the form $f(x) = x^3 + ax + b$.

We now introduce two new unknowns u and v and write $x = u + v$. Then on substituting for x, we get

$$\begin{aligned} f(x) = f(u + v) &= (u + v)^3 + a(u + v) + b \\ &= u^3 + 3u^2v + 3uv^2 + v^3 + au + av + b \\ &= u^3 + v^3 + (3uv + a)(u + v) + b. \end{aligned}$$

We now choose u and v so that $3uv + a = 0$, which removes the $u + v$ term. This gives us $v = -\dfrac{a}{3u}$. So if $f(x) = 0$, we get, in terms of u and v,

$$u^3 + v^3 + b = 0 = u^3 - \frac{a^3}{27u^3} + b.$$

If we multiply through by u^3, we get

$$u^6 + bu^3 - \frac{a^3}{27},$$

which we recognize as a quadratic equation in u^3. We solve for u^3 using the quadratic formula:

$$u^3 = \frac{-b \pm \sqrt{b^2 + (4a^3/27)}}{2}.$$

We get two solutions for u^3 and on taking cube roots, we obtain six solutions for u. Now the relation $v = -\dfrac{a}{3u}$ gives

$$v^3 = -\frac{a^3}{27u^3}.$$

If, say, we take

$$u^3 = \frac{-b + \sqrt{b^2 + (4a^3/27)}}{2},$$

then this leads to

$$v^3 = \frac{-b - \sqrt{b^2 + (4a^3/27)}}{2}.$$

This shows that in general u^3 and v^3 are what are called algebraic conjugate expressions. We can find a root $u+v$ of the original cubic by extracting cube roots in these expressions.

Let's try an example that Cardano gave in the *Ars magna*, namely, find the roots of $x^3 + 6x - 20 = 0$, which Cardano would have expressed as $x^3 + 6x = 20$, to avoid negative quantities. Here, $a = 6$, $b = -20$. Then

$$b^2 = 400, \quad \frac{4a^3}{27} = 32, b^2 + (4a^3/27) = 432 = (12)^2 \times 3.$$

Therefore, $u^3 = 10+6\sqrt{3}$, and, in principle we can extract cube roots to find three possible values for u (two of which involve complex cube roots of unity). From what we have said above, $v^3 = 10 - 6\sqrt{3}$ and thus a root of the cubic is

$$\sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}}.$$

This is a rather complicated expression. It is not so hard to show that it in fact equals 2, something that Cardano noted, but was not able to prove.

An interesting problem arises in the application of Cardano's method, which is quite difficult to appreciate. Cardano's formula involves the extraction of a square root and then of cube roots. This seems to be quite satisfactory, for it must be realized that when we talk of algebraic solutions of polynomial equations, we understand that the solutions should be given in terms of so called radicals–that is, expressions that involve extracting square, cube, and higher roots. Other types of solution may be given, in terms say of trigonometric functions, but these are taken to lie outside the domain of algebra. Now we will inevitably be drawn into the realm of complex numbers if the expression $b^2 + (4a^3/27)$ is negative, since we have to extract its square root. There are examples known where we have cubic polynomials whose three roots are all real and yet where Cardano's method leads inevitably to the use of complex number methods. This is not simply a defect of his method: it can be proved that in certain cases any expression for the roots in terms of radicals, even if the roots are real, must involve non-real expressions under the radical signs.

This became known as the *casus irreducibilis* or irreducible case of Cardano's method, and it has demonstrated the relevance and indeed inevitably of the use of complex numbers. Cardano's was aware of the problem but he was unable to develop the algebraic tools to clarify what is happening.

Another Italian, Rafael Bombelli (1526-72), wrote a textbook entitled *Algebra*, which gave a better exposition of Cardano's method. Parts of this book were published in 1572 just before his death. The *Algebra* followed the style of earlier books, such as Pacioli's *Summa de arithmetica*. It began with elementary material, illustrated with numerous problems. Bombelli had been influenced in his choice of problems by studying a Greek manuscript of the *Arithmetica* of Diophantus of Alexandria in the Vatican Library–an example of how, during the Renaissance, scholars began to rediscover the classical legacy of ancient Greece. The book progressed to the most modern work available in algebra at that time–the solution of cubic and quartic equations. Bombelli's *Algebra* remained popular into the 17th century and helped to teach the famous German mathematician Leibniz the rudiments of the subject.

Bombelli made use of a limited amount of algebraic notation, which enabled him to advance beyond the rhetorical style of the Islamic and earlier Italian mathematicians. He used *R.q.* to denote a square root, *R.c.* to denote a cube root, and so on. He also used parentheses to enclose large algebraic expressions and thus clarify meaning. He maintained the use of p for *plus* and m for *minus*, as in older Italian publications. (The $+$ and $-$ signs had already been introduced in German arithmetic books and these were subsequently universally adopted.) Bombelli's main notational innovation was the use of a semicircle under a number n to denote an nth power. So, for example,

$$1\overset{3}{\smile}\ p\ 6\overset{2}{\smile}\ m\ 3\overset{1}{\smile} \quad \text{denoted} \quad x^3+6x^2-3x.$$

Bombelli gave an enirely original treatment of the imaginary quantities that arise in the solution of cubic equations, in which he showed how such quantities can be manipulated like real quantities. He called the imaginary number $\sqrt{-1}$ *più di meno* (*plus of minus*) and $-\sqrt{-1}$ *meno di meno* (*minus of minus*). He wrote, for example, the imaginary number $3+2\sqrt{-1}$ as $3\,p\,di\,m\,2$ and $3-2\sqrt{-1}$ as $3\,m\,di\,m\,2$. He provided rules of multiplication for these imaginary quantities, such as

$$piu\,di\,meno \times piu\,di\,meno = meno$$

(meaning $\sqrt{-1}\times\sqrt{-1}=-1$), and gave examples to show how more complicated multiplication and division could be performed with imaginary quantities.

Let's look at Bombelli's treatment of a case of the cubic involving imaginary quantities. Take the polynomial $x^3-15x-4$. This has the integer root $x=4$, and it's easy to see that all three roots of this polynomial are real. In fact, the polynomial factorizes as $(x-4)(x^2+4x+1)$ and so the other two roots can be found in terms of real square roots. If we simply stick to Cardano's method of solution, we see that $b^2+(4a^3/27) = -484$ and so we must use imaginary quantities. We get a root

$$\sqrt[3]{2+11\sqrt{-1}} + \sqrt[3]{2-11\sqrt{-1}},$$

which is actually real. Without using this precise expression, Bombelli noticed that the two expressions inside the cube root signs are complex conjugate numbers (that is, of the form $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$), and he surmised that the cube roots must also be complex conjugates (which is correct). Writing then these conjugate cube roots as $c+d\sqrt{-1}$ and $c-d\sqrt{-1}$, and using the fact that 4 is a root, we get

$$4 = c+d\sqrt{-1}+c-d\sqrt{-1}$$

and therefore $c = 2$. As we also have

$$(c+d\sqrt{-1})^3 = 2+11\sqrt{-1},$$

we obtain on cubing, since $c = 2$,

$$8+12d\sqrt{-1}-6d^2-d^3\sqrt{-1} = 2+11\sqrt{-1}$$

and thus comparing real and imaginary parts, $8-6d^2 = 2$, so that we can take $d = 1$. In this way, Bombelli had shown that the cube roots of imaginary quantities could be expressed as imaginary quantities. Admittedly, he had to know the real root $x = 4$, and he used a certain amount of trial and error, but his approach seems surprisingly modern. In any case, the imaginary numbers had shown their value. Sometimes we can make a well informed guess as to the cube root of a imaginary quantity and then obtain straightforward real solutions of cubic equations. It might be added that there exist more complicated examples of the *casus irreducibilis* where there are no simple formulae for real roots of cubics, except by using imaginary numbers. In such cases, solutions expressed in terms of sines of angles also exist, as we will explain later.

We have seen how the roots of a cubic polynomial may be found by what has become known as Cardano's method. The method for calculating the roots of a quartic polynomial (one of degree 4) is due to Ludovico Ferrari (1522-65). Ferrari was born in Bologna. At the age of fourteen, he was sent to Milan to work with Cardano, from whom he learned mathematics. Ferrari proved to be mathematically gifted and he assisted Cardano in his research. In answer to a question posed to Cardano, Ferrari discovered a method for solving a specific quartic polynomial which proved to be applicable to all cases. The question was this:

divide 10 into three parts in continued proportion so that the product of the first two parts is 6.

This means that we must find three real numbers a, b and c so that $a \leq b \leq c$ and

$$a + b + c = 10$$
$$ab = 6.$$

We express the fact that the numbers are in *continued proportion* by setting

$$\frac{c}{b} = \frac{b}{a}.$$

This last property translates into $b^2 = ac$. Now we obtain

$$b^3 = abc = 6c = 6(10 - a - b)$$

and since $a = 6/b$, this gives

$$b^3 = 6(10 - \frac{6}{b} - b).$$

Multiplying by b, we obtain $b^4 = 60b - 36 - 6b^2$ and this translates, in our notation, to a quartic polynomial in b, where the coefficient of b^3 is 0, namely:

$$b^4 + 6b^2 - 60b + 36 = 0.$$

Ferrari solved this by using the technique of completing the square, and introducing what has become known as an *auxiliary cubic*, which of course could be solved by Cardano's method. The fundamental features of his method remain unchanged to this day, and it is remarkable that such good progress could be made with the disadvantage of poor notation. A simple algebraic identity is also used:

$$(x^2 + c + d)^2 = (x^2 + c)^2 + 2d(x^2 + c) + d^2,$$

which is well known to us, but which was expressed geometrically by Ferrari.

We continue the solution in modern notation, although the essence of Ferrari's solution is maintained. We rearrange to get

$$b^4 + 6b^2 = 60b - 36.$$

Take half the coefficient of b^2, square it to get 9, and add this quantity to both sides:

$$b^4 + 6b^2 + 9 = 60b - 27.$$

The left hand side is now the perfect square $(b^2 + 3)^2$–remember that this trick is used to solve a quadratic equation. Therefore,

$$(b^2 + 3)^2 = 60b - 27.$$

The left hand side is a perfect square and we want to try to add something to the right hand side to obtain a perfect square, while maintaining one on the left. By our quadratic identity above, we have for any number d,

$$\begin{aligned}(b^2+3+d)^2 &= (b^2+3)^2 + 2d(b^2+3) + d^2\\ &= 60b - 27 + 2d(b^2+3) + d^2\\ &= 2db^2 + 60b + 6d + d^2 - 27.\end{aligned}$$

The expression on the right above can be interpreted as a quadratic in b with variable coefficients. Now recall from the theory of quadratic equations that a quadratic polynomial ax^2+bx+c is expressible as the perfect square

$$(\sqrt{a}x + \sqrt{c})^2$$

precisely when $b^2 = 4ac$ (this is the condition for the quadratic to have equal roots). So, we can arrange for the quadratic in b on the right above to be a perfect square if

$$60^2 = 8d(6d + d^2 - 27)$$

and this translates into

$$450 = d^3 + 6d^2 - 27d.$$

Therefore, we need d to be a solution of the cubic equation

$$d^3 + 6d^2 - 27d - 450 = 0.$$

We can solve this by Cardano's method by first removing the d^2 term by a linear change of variables. Once a value for d is found, we get an equation

$$(b^2+3+d)^2 = \left(\sqrt{2d}b + \sqrt{6d+d^2-27}\right)^2,$$

which we can solve by taking square roots and then applying the quadratic formula. (Of course the actual value b is going to be rather messy, so there is no simple solution to the original problem of dividing 10 into three numbers with the special properties.)

Ferrari's solution was presented to the world by Cardano in the *Ars magna* of 1545. Remember that Cardano had previously sworn an oath to Tartaglia that he would not divulge the secret of solving a cubic equation. It seems however that Cardano became aware that Tartaglia did not have priority in the discovery. Cardano and Ferrari travelled to Bologna to inspect the working papers of Scipione del Ferro, which had been preserved at the university there. On confirming that del Ferro already had the solution of the cubic before Tartaglia, Cardano felt that the force of the oath no longer applied and that he was justified to publish his solution of the cubic equation. Tartaglia responded by accusing

Cardano of bad faith and duplicity. The ensuing quarrel between the two sides was largely conducted by Ferrari and Tartaglia in a series of letters, which were printed and published, and are still available to this day. To try to settle the quarrel, Ferrari and Tartaglia met in a public disputation held in Milan in August 1548. It seems that Ferrari was held to be the victor. Mathematics rarely attracts such publicity or arouses such bad feelings!

It was later shown by Bombelli that the basic features of Ferrari's method are applicable to finding the roots of any quartic polynomial, as we will explain here. Suppose that we require the roots of the quartic

$$x^4 + ax^3 + bx^2 + cx + d.$$

As in the cubic case, we substitute $x = y - r$ into the x-expression and choose r so that the coefficient of y^3 in the resulting quartic in y is 0. This is done by setting

$$r = \frac{a}{4}.$$

It suffices, therefore, to assume that our quartic is

$$x^4 + ax^2 + bx + c.$$

Let α be a root. Then

$$\alpha^4 + a\alpha^2 = -b\alpha - c.$$

Complete the square on the left to get

$$\begin{aligned} \alpha^4 + a\alpha^2 + \frac{a^2}{4} &= \frac{a^2}{4} - b\alpha - c \\ \left(\alpha^2 + \frac{a}{2}\right)^2 &= \frac{a^2}{4} - b\alpha - c. \end{aligned}$$

As before, we look for d so that

$$\begin{aligned} \left(\alpha^2 + \frac{a}{2} + d\right)^2 &= \left(\alpha^2 + \frac{a}{2}\right)^2 + 2d\left(\alpha^2 + \frac{a}{2}\right) + d^2 \\ &= \frac{a^2}{4} - b\alpha - c + 2d\alpha^2 + d\alpha + d^2 \\ &= 2d\alpha^2 - b\alpha + \frac{a^2}{4} - c + da + d^2. \end{aligned}$$

The expression in α on the right hand side above is a perfect square in α provided

$$b^2 = 8d\left(d^2 + da + \frac{a^2}{4} - c\right)$$

and this again gives a cubic in d which we can solve. Once we have a value for d, we can find α by taking square roots and applying the quadratic formula.

We see therefore that by the middle of the 16th century, considerable progress had been made on the problem of finding the roots of polynomials. Despite poor notation, the fundamental principles had been laid down for finding the roots of cubic and quartic polynomials. The roots were expressed in terms of formulae that involve the square roots and cube roots of possibly imaginary magnitudes. This was entirely reasonable, as even to give the roots of polynomials such as $x^3 + c$ or $x^4 + d$ involves the extraction of cube roots and repeated square roots. This advance may be considered the most significant mathematical step made since the algebraic work of Diophantus in the third century. No further progress was made in finding similar expressions to describe the roots of quintic polynomials (polynomials of degree 5). The hope was clearly that the roots might be expressed in terms of radicals involving square, cube and fifth roots, but no amount of ingenuity could give an answer except in very special cases.